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Nissim Francez: *Proof-theoretic Semantics*
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During the second half of the twentieth century, most of logic bifurcated into model theory and proof theory. Model theory, as established by Tarski & Co., was considered as a matter of "semantics": it investigated the relationship between formal languages and the domains of entities about which the languages were supposed to be. Proof theory, on the other hand, was taken to be a matter of "syntax": not concerning what the formulas of formal languages are about, but about certain relations among them. Hence, when there appeared the term *proof-theoretic semantics* (PTS), it sounded quite paradoxical: how could there be a "syntactic semantics"?

The solution to this quandary lies, I am convinced, in the elucidation of the misleading role the term "syntax" has played within modern logic (and philosophy). Primarily, syntax is a theory of "well-formedness"—of the delimitation of the range of expressions which make up a given language. In this sense, syntax indeed has nothing to do with semantics and it would be futile to try to base a semantics on it. However, the term "syntax" has also been used to refer to inferences, derivations and proofs, and if considered in *this* sense, it is no longer so clear that it is unrelated to semantics. Indeed, the second half of the twentieth century also witnessed the rise of the so-called use-theories of meaning, at least some of which identified meaning of an expression specifically with the role conferred on it by the inferential rules governing its proper employment.[1]

Proof-theoretic semantics is also closely connected with the search for the proper semantics of intuitionistic logic. While classical logic has the natural truth-functional semantics, there was, for some time, no such canonical semantics for intuitionist logic. What was subsequently to become accepted as its adequate semantic account was its so-called BHK-interpretation (see Troelstra & van Dalen 1988): the idea that the semantics is based on the concept of proof. This idea is usually incorporated into PTS in such a way that the meaning of a sentence is considered as the set of all its proofs (or all its "canonical" proofs) and that logical connectives express ways to combine proofs of components into proofs of a compound. The term "proof-theoretic semantics" was introduced by Schroeder-Heister

[1] See Peregrin (2006a) for a discussion.

(1991) and its development is often associated with Prawitz (2006). Francez's book presents its elaboration not only for the formal languages of logic, but also for natural languages.

PTS built on this basis seems to pose two problems. Firstly, the association of the meaning of a statement with the set of its proofs appears to be epistemologically unrealistic: do we want to say that whoever understands a sentence is bound to know all the ways to prove it? And secondly, even if we accept this, the theory gives us meanings of sentences, but what about those of sub-sentential expressions?

Compare the situation with the well-known origin of model-theoretic semantics (MTS) for the languages of logic and its extension to natural language initiated by Montague (1974). Here, the starting point is the Frege's (1891) idea concerning explicating meanings of predicative expressions as functions from objects to truth values, which led to the standard truth-functional semantics for logical operators. This then led on to the general idea that the meanings of expressions of all other categories, save sentences and names, are functions built on the basis of the denotations of sentences (truth values) and names (objects). (For a very general formal language this was proposed, for the first time, by Church 1940.) Montague then provided his elegant semantic treatment of a fragment of English, which secured MTS a place on a philosophical pedestal.

But note an important feature of the Montagovian MTS: it was not extensional because the denotations of sentences were not their truth values; they were rather functions from possible worlds to truth values (or the sets of possible worlds that the functions characterize). Despite this, logical connectives could, in effect, retain their truth-functional denotations. Thus, even if we were to explicate the denotations of sentences as the sets of their proofs, it need not follow that if PTS were to follow in Montague's footsteps, the denotations of logical connectives would have to be something as monstrous as functions from pairs of sets of proofs to sets of proofs. And indeed it is the BHK-interpretation that indicates to what such denotations could be reduced: to methods of combining proofs that could be quite simple.

In view of this, we can put, and this is something Francez shows very explicitly in the book, the whole Frege-Church-Montague functional machinery into the services of PTS. On the level of propositional logic, we have just to assume that we have denotations of statements (the sets of their proofs) and derive the denotations of sentential operators as corresponding functions. Then we can try to reduce them to something simpler; in the case of conjunction it could, for example, be combining proofs of the two conjuncts into the proof of the conjunction, which could be nothing more complicated than putting the two proofs beside each other. Thus, the meaning of "$\wedge$", for example, may be the function which maps two sets of proofs, P_1 and P_2, on the set of proofs that contains, for every proof D_1 of a formula A_1 that belongs to P_1 and every proof D_2 of a formula A_2 that belongs to P_2, the proof

$$\frac{D_1 \qquad D_2}{A_1 \wedge A_2}$$

The situation is a little more complicated on the level of predicate logic, where Church made use of one more category the denotations of which were primitive, namely names (which, according to him, denote individuals—elements of a universe). Here Francez's approach diverges from the model-theoretic one: the category of expressions he chooses as the other primitive one are (individual) variables. A variable, according to him, denotes itself. As a result, a quantifier takes the denotation of a sentence (the set of its proofs) plus that of a variable (the variable itself) to the denotation of a quantified sentence. This, of course, presupposes that the category of sentences includes open formulas. In this way, the meaning of "$\forall$", for example, is a function which maps a set of proofs P and a variable v on the set of proofs that contains, for every proof D of a formula A that belongs to P (and such that v is not in any premise or undischarged assumption of D), the proof

$$\frac{D}{\forall v A}$$

It turns out, however, that this version of PTS has a property that may be seen as problematic: the meanings of sentences, *viz.* the sets of their canonical proofs, turn out to be overly fine-grained. (For example, the meaning of $A_1 \wedge A_2$ comes out as different from that of $A_2 \wedge A_1$. This may make some sense for a natural language, but much less for a logical language in which the two sentences are provably equivalent and provably intersubstitutive w.r.t. logical equivalence.) Hence it would seem that what would fare better in this respect would be the identification of the meaning of a sentence with the set of *grounds* of the sentence: sets of all sets of formulas from which the formula is derivable. (Also we might think about including only *maximal* grounds, which would then be not so far from possible worlds, and PTS would come slightly closer to MTS.) It is a pity that Francez does not elaborate on this idea.

Francez uses the concept of ground also for the definition of proof-theoretical consequence: *A* is a consequence of *X* iff everything that is a ground for *X* is a ground for *A*. Again, it seems to me to be a pity that Francez does not tell us more about the proof-theoretical relation of consequence defined in this way. (Usually it is noted that there is a gap between *derivability*, as a proof-theoretical matter, and *consequence*, which must be defined model-theoretically (see, e.g., Etchemendy 1990). Carnap (1934) tried to account for this gap in purely proof-theoretic terms (see Peregrin 2014, Ch. 7); and it would be nice to learn what ambitions Francez has using his definition.)

The second part of Francez's book applies PTS to natural language, thus creating an antipode to the Montagovian MTS. Some of the ideas already embodied into PTS for the languages of logic can be straightforwardly transferred to natural languages, but in some respects natural languages are different. In particular, we can treat at least some of the connectives on a par with their logical counterparts; but the way quantification operates in natural language is very different from the standard Fregean quantifiers embraced by logic.

Francez, nevertheless, approaches the situation analogously to that of the formal languages. He enriches the fragment of natural language by using "individual parameters", which play a role somewhat analogous to that played by variables in formal languages. (Thus the whole language he works with is comprised of natural language plus "open" sentences that can be assembled from elements of natural language and parameters.) And though the mechanism of quantification is different, Francez's way of coping with it proof-theoretically is quite similar: the proof of a general statement builds on the proof of the corresponding statement with an indeterminate individual parameter.

One of the crucial features of Montagovian formal semantics was that it accounted for intensional contexts, that by engaging possible worlds it surpassed the limits of extensional semantics (see Peregrin 2006b). The proof-theoretic account of Francez has no lesser ambitions: it, too, aspires to account for the intensionality of natural language. However, here the method differs greatly from the model-theoretic one. What does the work here is nothing like possible worlds. Francez introduces a new kind of individual parameters, which he calls *notional parameters*. These parameters have inferential properties different from ordinary individual parameters. For example, while *John finds a unicorn* is introduced on the basis of *John finds x* and *x is a unicorn* (hence it follows that there is something that is a unicorn), the grounds of the introduction of *John seeks a unicorn* are different: they are *John seeks n* and *n is being a unicorn* (where *n* is a notional parameter) and it has no existential import.

Francez's book is literally packed with information; it is, in fact, multiple books in one. It contains a concise introduction into Gentzenian proof theory; it contains an elaboration of the semantic ideas of both Gentzen and the BHK-people, taking them forward into an explicit theory of semantic values; and it contains—and this is the most original part—also an elaborated sketch of PTS for a fragment of natural language, parallel to the celebrated MTS of Montague. Thus it shows that proof-theory is not syntax—at least not in any sense that would prevent it from conferring meanings on expressions. In this way it is, aside of presenting a wealth of new results, usable also as a handbook of structural proof theory. And given that College Publications, who published the book, do not overcharge their customers, buying it is a true deal!

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